

Twisted modules and quasi-modules for vertex operator algebras

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To James Lepowsky and Robert Wilson with admiration and appreciation.

ABSTRACT. We use a result of Barron, Dong and Mason to give a natural isomorphism between the category of twisted modules and the category of quasi-modules of a certain type for a general vertex operator algebra.

1. Introduction

In the theory of vertex operator algebras, for a vertex operator algebra V , in addition to the notion of V -module one has the notion of σ -twisted V -module where σ is a finite order automorphism of V . For a V -module W , each element $v \in V$ is represented by a vertex operator

$$Y_W(v, x) \in \text{Hom}(W, W((x))) \subset (\text{End } W)[[x, x^{-1}]],$$

where these vertex operators are mutually local in the sense that for $u, v \in V$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k [Y_W(u, x_1), Y_W(v, x_2)] = 0.$$

Twisted modules were first introduced and used by Frenkel, Lepowsky and Meurman in their construction of the moonshine module vertex operator algebra V^\natural (see [L1], [FLM]). Let V be a vertex operator algebra and let σ be an automorphism of order N . For a σ -twisted V -module W ([L1], [FLM], [FFR], [D]), each element v of V is represented by a twisted vertex operator

$$Y_W(v, x) \in \text{Hom}(W, W((x^{1/N}))) \subset (\text{End } W)[[x^{1/N}, x^{-1/N}]],$$

where these twisted vertex operators are also mutually local.

In a recent work [Li3], to associate certain (infinite-dimensional) Lie algebras with vertex algebras, we studied what we called quasi local vertex operators (cf. [GKK]). Let W be any vector space. A subset S of $\text{Hom}(W, W((x)))$ is said to be quasi local if for any $a(x), b(x) \in S$ there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2)b(x_2)a(x_1).$$

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It was proved therein that any maximal quasi local subspace has a natural vertex algebra structure and any quasi local subset generates a vertex algebra. This particular result generalizes the main result of [Li1], which states that for any vector space W , any set of mutually local vertex operators on W generates a vertex algebra with W as a natural module. However, the space W under the natural action is not a module for vertex algebras generated by quasi local vertex operators on W , though a certain weaker version of Jacobi identity was proved to hold. This motivated us to introduce a new notion of quasi module for a vertex algebra. For a quasi module W for a vertex algebra V , each element v of V is represented by a vertex operator

$$Y_W(v, x) \in \text{Hom}(W, W((x))) \subset (\text{End } W)[[x, x^{-1}]]$$

and the vertex operators $Y_W(v, x)$ for $v \in V$ form a quasi local subspace.

On twisted modules for vertex operator algebras there is a conceptual work [BDM], in which for any vertex operator algebra V and for any positive integer k , a canonical isomorphism was established between the category of V -modules and the category of twisted modules for the tensor product vertex operator algebra $V^{\otimes k}$ with respect to permutation automorphisms. In [BDM], a central role was played by the geometric change-of-coordinate $x = z^k$. It has been known ([Z], [H1-3], cf. [L2]) that for any vertex operator algebra V and for any $f(z) \in z\mathbb{C}[[z]]$ with $f'(0) \neq 0$, the change-of-coordinate $x = f(z)$ gives rise to a “new” vertex operator algebra structure on V , which was proved to be isomorphic to V . A special change-of-coordinate played a very important role in the study of modular invariance of graded characters ([Z], [DLM2]).

It has been well known (cf. [FZ]) that (untwisted) affine Lie algebras together with their highest weight modules can be naturally associated with vertex operator algebras and their modules. Furthermore, twisted affine Lie algebras together with their highest weight modules (see [K]) can be associated with twisted modules for those vertex operator algebras (cf. [FLM], [Li2]). On the other hand, it was proved in [Li3] that twisted affine Lie algebras, which are represented in a different form, together with their highest weight modules, can be naturally associated with quasi-modules for the vertex operator algebras associated with the untwisted affine Lie algebras. This suggests that there exist a natural connection between twisted modules and quasi modules for a general vertex operator algebra.

The main purpose of this paper is to give a natural connection between twisted modules and quasi-modules for a general vertex operator algebra. Indeed, the goal has been achieved by using [BDM], thanks to Barron, Dong and Mason for their beautiful work. What we have proved is that the same change-of-coordinates, used by Barron, Dong and Mason, give rise to a natural isomorphism between the category of twisted modules and the category of quasi-modules of a certain special type.

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2. Twisted modules and quasi-modules

We here present the main result, a natural isomorphism between the category of twisted modules and the category of quasi-modules of a certain type for a general vertex operator algebra.

First, we recall the definitions of the notions of twisted module and quasi-module. Let $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ be a vertex operator algebra, fixed throughout this section. For the definition and basic properties we refer to [FLM] and [FHL]. Let σ be an automorphism of V of order N (a positive integer). Then $V = \coprod_{j=0}^{N-1} V^j$, where $V^j = \{u \in V \mid \sigma(u) = \omega_N^j u\}$ and $\omega_N = \exp(2\pi\sqrt{-1}/N)$, the principal primitive N -th root of unity.

A *weak σ -twisted V -module* (cf. [L1], [FLM], [FFR], [D], [DLM1]) is a vector space W equipped with a linear map

$$(2.1) \quad \begin{aligned} Y_W : \quad V &\rightarrow \text{Hom}(W, W((x^{\frac{1}{N}}))) \subset (\text{End } W)[[x^{\frac{1}{N}}, x^{-\frac{1}{N}}]] \\ v &\mapsto Y_W(v, x) \end{aligned}$$

such that

$$Y_W(\mathbf{1}, x) = 1_W \text{ (the identity operator on } W),$$

and for $u, v \in V$,

$$(2.2) \quad \begin{aligned} &x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_W(u, x_1) \\ &= \frac{1}{N} \sum_{r=0}^{N-1} x_1^{-1} \delta \left(\omega_N^r \left(\frac{x_2 + x_0}{x_1} \right)^{\frac{1}{N}} \right) Y_W(Y(\sigma^r u, x_0) v, x_2) \end{aligned}$$

(the *twisted Jacobi identity*). Note that as a convention, for $\alpha \in \mathbb{R}$, the expressions $(x_1 \pm x_2)^\alpha$ are understood as the formal series in the nonnegative integral powers of the second variable x_2 . That is,

$$(x_1 \pm x_2)^\alpha = \sum_{i \in \mathbb{N}} \binom{\alpha}{i} (\pm 1)^i x_1^{\alpha-i} x_2^i \in x_1^\alpha \mathbb{R}[x_1^{-1}][[x_2]].$$

If $u \in V^j$ with $0 \leq j \leq N-1$, the twisted Jacobi identity becomes

$$(2.3) \quad \begin{aligned} &x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_W(u, x_1) \\ &= x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) \left(\frac{x_2 + x_0}{x_1} \right)^{-\frac{j}{N}} Y_W(Y(u, x_0) v, x_2). \end{aligned}$$

By taking $v = \mathbf{1}$, one obtains

$$(2.4) \quad Y_W(u, x) \in x^{\frac{j}{N}} \text{Hom}(W, W((x))).$$

This particular property amounts to

$$(2.5) \quad Y_W(\sigma u, x) = \lim_{x^{1/N} \rightarrow \omega_N x^{1/N}} Y_W(u, x).$$

REMARK 2.1. Note that the above defined notion of σ -twisted V -module, which is the one defined in [DLM2] and [BDM], corresponds to the notion of σ^{-1} -twisted V -module in [DLM1].

The twisted Jacobi identity is equivalent to the following *weak commutativity and associativity* ([DL], [Li2]): For $u, v \in V$, there exists a nonnegative integer k such that

$$(2.6) \quad (x_1 - x_2)^k [Y_W(u, x_1), Y_W(v, x_2)] = 0,$$

and for $u \in V^j$, $v \in V$, $w \in W$, $0 \leq j \leq N-1$, there exists a nonnegative integer l such that

$$(2.7) \quad \begin{aligned} & (x_0 + x_2)^{l-j/N} Y_W(u, x_0 + x_2) Y_W(v, x_2) w \\ &= (x_2 + x_0)^{l-j/N} Y_W(Y(u, x_0)v, x_2) w. \end{aligned}$$

From now on we fix an automorphism σ of order N for the fixed vertex operator algebra V . Set

$$(2.8) \quad G = \langle \sigma \rangle \subset \text{Aut}(V) \text{ (the full automorphism group of } V \text{)}.$$

Let $\phi : G \rightarrow \mathbb{C}^\times$ be the (injective) group homomorphism defined by $\phi(\sigma^j) = \omega_N^j$ for $j = 0, \dots, N-1$.

A *quasi V -module* [Li3] is a vector space W equipped with a linear map

$$\begin{aligned} Y_W : \quad V &\rightarrow \text{Hom}(W, W((x))) \subset (\text{End } W)[[x, x^{-1}]], \\ v &\rightarrow Y_W(v, x) \end{aligned}$$

such that

$$Y_W(\mathbf{1}, x) = 1_W,$$

and for $u, v \in V$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$(2.9) \quad \begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) \\ & \quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) p(x_1, x_2) Y_W(v, x_2) Y_W(u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) p(x_1, x_2) Y_W(Y(u, x_0)v, x_2) \end{aligned}$$

(the *quasi Jacobi identity*).

DEFINITION 2.2. A quasi V -module (W, Y_W) is called a (G, ϕ) -*quasi V -module* if for any $u, v \in V$, there exists a nonnegative integer k such that (2.9) holds with $p(x_1, x_2) = (x_1^N - x_2^N)^k$ and such that

$$(2.10) \quad Y_W(\phi(g)^{-L(0)} g(u), x) = Y_W(u, \phi(g)x) \quad \text{for } g \in G, u \in V.$$

Follow [BDM] to define $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}_+$ by

$$(2.11) \quad \exp \left(- \sum_{n \in \mathbb{Z}_+} a_n x^{n+1} \frac{d}{dx} \right) \cdot x = \frac{1}{N} (1+x)^N - \frac{1}{N},$$

where \mathbb{Z}_+ denotes the set of positive integers, and then set

$$(2.12) \quad \Delta_N(x) = \exp \left(\sum_{n \in \mathbb{Z}_+} a_n x^{-n/N} L(n) \right) N^{-L(0)} x^{(1/N-1)L(0)},$$

an invertible element of $(\text{End } V)[[x^{1/N}, x^{-1/N}]]$. We have

$$(2.13) \quad \Delta_N(x^N)^{-1} = (Nx^{N-1})^{L(0)} \exp \left(- \sum_{n \in \mathbb{Z}_+} a_n x^{-n} L(n) \right).$$

As in [BDM] we shall also heavily use the expression $\Delta_N(x^N)^{-1}$. For convenience we set

$$(2.14) \quad \Phi(x) = \Delta_N(x^N)^{-1} \in \text{Hom}(V, V[x, x^{-1}]).$$

The following result was proved in [BDM]:

PROPOSITION 2.3. *For any $u \in V$, we have*

$$(2.15) \quad \Delta_N(x)Y(u, x_0)\Delta_N(x)^{-1} = Y(\Delta_N(x+x_0)u, (x+x_0)^{1/N} - x^{1/N})$$

in $(\text{End } V)[[x_0^{\pm 1}, x^{\pm \frac{1}{N}}]]$, and

$$(2.16) \quad \Phi(x)Y(u, x_0) = Y(\Phi(x+x_0)u, (x+x_0)^N - x^N)\Phi(x)$$

in $(\text{End } V)[[x_0^{\pm 1}, x^{\pm 1}]]$.

REMARK 2.4. Recall from [BDM] the interpretation of the formal variable notations in Proposition 2.3. First, for any nonzero $\alpha \in \frac{1}{N}\mathbb{Z}$, under the convention we have

$$(x+x_0)^\alpha - x^\alpha = \sum_{i \in \mathbb{Z}_+} \binom{\alpha}{i} x^{\alpha-i} x_0^i = x^{\alpha-1} x_0 (\alpha + x_0 f),$$

where $f = \sum_{i \geq 2} \binom{\alpha}{i} x^{1-i} x_0^{i-2} \in \mathbb{R}[x^{-1}][[x_0]]$. For $n \in \mathbb{Z}$, it is understood that

$$((x+x_0)^\alpha - x^\alpha)^n = x^{n(\alpha-1)} x_0^n \sum_{i \in \mathbb{N}} \binom{n}{i} \alpha^{n-i} x_0^i f^i \in x^{n\alpha} x_0^n \mathbb{R}[x, x^{-1}][[x_0]].$$

Then for $u, v \in V$, we have

$$Y(u, (x+x_0)^\alpha - x^\alpha)v = \sum_{m \in \mathbb{Z}} u_m v ((x+x_0)^\alpha - x^\alpha)^{-m-1} \in V[x^{\pm 1/N}][[(x_0)]].$$

This explains the formal variable notations in Proposition 2.3. As we shall mention in the following Remark, we shall also use another (different) substitution. For this purpose, we also write

$$Y(u, z)|_{z=(x+x_0)^\alpha - x^\alpha, x > x_0}$$

for this particularly defined expression $Y(u, (x+x_0)^\alpha - x^\alpha)$. It was showed in [BDM], page 363, that for $h, \alpha \in \frac{1}{N}\mathbb{Z}$,

$$(2.17) \quad (x_2^\alpha + z_0)^h|_{z_0=(x_2+x_0)^\alpha - x_2^\alpha, x_2 > x_0} = (x_2 + x_0)^{\alpha h}.$$

Warning: The following expansion

$$\begin{aligned} (z - x^\alpha)^n|_{z=(x+x_0)^\alpha} &= \sum_{i \in \mathbb{N}} (-1)^i \binom{n}{i} (x+x_0)^{\alpha(n-i)} x^{\alpha i} \\ &= \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} (-1)^i \binom{n}{i} \binom{(n-i)\alpha}{j} x^{n\alpha-j} x_0^j \end{aligned}$$

is an infinite divergent sum if $n < 0$.

REMARK 2.5. We shall need a different substitution $z = (x + x_0)^N - x^N$ for rational powers z^α , $\alpha \in \frac{1}{N}\mathbb{Z}$. Let $p(x_0, x) = x_0^k + xq(x_0, x)$, where k is a positive integer and $q(x_0, x)$ is a polynomial. We consider the following expansion

$$p(x_0, x)^\alpha = (x_0^k + xq(x_0, x))^\alpha = \sum_{i \in \mathbb{N}} \binom{\alpha}{i} x_0^{k(\alpha-i)} x^i q(x_0, x)^i \in \mathbb{R}[x_0, x_0^{-1}][[x]],$$

and we use the notation

$$z^\alpha|_{z=p(x_0, x), x_0 > x}$$

for this particular expansion. That is,

$$z^\alpha|_{z=p(x_0, x), x_0 > x} = (x_0^k + y)^\alpha|_{y=xq(x_0, x)}$$

under the usual convention. In particular, for $p(x_0, x) = x_0 + x$ we have

$$z^\alpha|_{z=x_0+x, x_0 > x} = (x_0 + x)^\alpha.$$

For any $h(x_0, x) \in \mathbb{R}[x_0, x]$ we have

$$\begin{aligned} & z^\alpha|_{z=p(x_0, x)+xh(x_0, x), x_0 > x} \\ &= \sum_{i \in \mathbb{N}} \binom{\alpha}{i} x_0^{k(\alpha-i)} (xq(x_0, x) + xh(x_0, x))^i \\ &= \sum_{i, j \in \mathbb{N}} \binom{\alpha}{i} \binom{i}{j} x_0^{k(\alpha-i)} (xq(x_0 + x))^{i-j} (xh(x_0, x))^j \\ &= \sum_{r, j \in \mathbb{N}} \binom{\alpha}{r+j} \binom{r+j}{j} x_0^{k(\alpha-r-j)} (xq(x_0 + x))^r (xh(x_0, x))^j \\ &= \sum_{r \in \mathbb{N}} \sum_{j \in \mathbb{N}} \binom{\alpha}{j} \binom{\alpha-j}{r} x_0^{k(\alpha-r-j)} (xq(x_0 + x))^r (xh(x_0, x))^j \\ &= \sum_{j \in \mathbb{N}} \binom{\alpha}{j} z_0^{\alpha-j}|_{z_0=p(x_0, x), x_0 > x} (xh(x_0, x))^j \\ (2.18) \quad &= (z_0 + xh(x_0, x))^\alpha|_{z_0=p(x_0, x), x_0 > x}. \end{aligned}$$

Using this and (2.17) we get

$$\begin{aligned} (z_0 + x^N)^\alpha|_{z_0=(x_0+x)^N-x^N, x_0 > x} &= z^\alpha|_{z=(x_0+x)^N, x_0 > x} \\ &= (x_0^N + y)^\alpha|_{y=(x_0+x)^N-x_0^N, x_0 > x} \\ (2.19) \quad &= (x_0 + x)^{N\alpha}. \end{aligned}$$

We shall need the following simple result:

LEMMA 2.6. For any $\alpha \in \mathbb{C}^\times$,

$$(2.20) \quad \Phi(x)\alpha^{-L(0)} = \alpha^{-NL(0)}\Phi(\alpha x),$$

$$(2.21) \quad \alpha^{L(0)}\Delta_N(x) = \lim_{x^{1/N} \rightarrow \alpha x^{1/N}} \Delta_N(x)\alpha^{NL(0)}$$

hold in $\text{Hom}(V, V[x, x^{-1}])$ and in $\text{Hom}(V, V[x^{\frac{1}{N}}, x^{-\frac{1}{N}}])$, respectively.

PROOF. We shall just prove the first identity, as the second will follow easily. Since $[L(0), L(n)] = -nL(n)$ for $n \in \mathbb{Z}$, it follows that

$$\alpha^{L(0)}L(n)\alpha^{-L(0)} = \alpha^{-n}L(n).$$

Thus

$$\begin{aligned}\alpha^{L(0)} \left(\sum_{n \in \mathbb{Z}_+} a_n x^{-n} L(n) \right) \alpha^{-L(0)} &= \sum_{n \in \mathbb{Z}_+} \alpha^{-n} a_n x^{-n} L(n) \\ &= \sum_{n \in \mathbb{Z}_+} a_n (\alpha x)^{-n} L(n).\end{aligned}$$

Using this we obtain

$$\begin{aligned}\Phi(x) \alpha^{-L(0)} &= (Nx^{N-1})^{L(0)} \alpha^{-L(0)} \alpha^{L(0)} \exp \left(- \sum_{n \in \mathbb{Z}_+} a_n x^{-n} L(n) \right) \alpha^{-L(0)} \\ &= \alpha^{-NL(0)} (N(\alpha x)^{N-1})^{L(0)} \exp \left(- \sum_{n \in \mathbb{Z}_+} a_n (\alpha x)^{-n} L(n) \right) \\ &= \alpha^{-NL(0)} \Phi(\alpha x),\end{aligned}$$

proving the assertion. \square

The following is the first half of our main result of this paper:

THEOREM 2.7. *Let (W, Y_W) be a weak σ -twisted V -module. For $u \in V$, as in [BDM] set*

$$(2.22) \quad \tilde{Y}_W(u, x) = Y_W(\Phi(x)u, x^N) \in (\text{End } W)[[x, x^{-1}]].$$

Then (W, \tilde{Y}_W) carries the structure of a (G, ϕ) -quasi V -module.

PROOF. First, for $u \in V$, as $\Phi(x)u \in V[x, x^{-1}]$, we have

$$\tilde{Y}_W(u, x) \in \text{Hom}(W, W((x))).$$

Second, we have

$$\tilde{Y}_W(\mathbf{1}, x) = Y_W(\mathbf{1}, x^N) = 1_W,$$

as $\Phi(x)\mathbf{1} = \mathbf{1}$, which is due to the fact that $L(n)\mathbf{1} = 0$ for $n \geq 0$.

Third, from Lemma 2.6, for any N -th root of unity α , we have

$$\Phi(x) \alpha^{-L(0)} = \Phi(\alpha x).$$

For $g \in G$, $u \in V$, we have

$$\begin{aligned}\tilde{Y}_W(\phi(g)^{-L(0)} g(u), x) &= Y_W(\Phi(x) \phi(g)^{-L(0)} g(u), x^N) \\ &= Y_W(\Phi(\phi(g)x) g(u), x^N) \\ &= \lim_{z \rightarrow \phi(g)x} Y_W(\Phi(\phi(g)x) u, z^N) \\ &= \tilde{Y}_W(u, \phi(g)x),\end{aligned}$$

noticing that $g\Phi(x) = \Phi(x)g$.

Now it remains to prove the quasi Jacobi identity. Let $u, v \in V$. Since $\Phi(x)u, \Phi(x)v \in V[x, x^{-1}]$, there exists a nonnegative integer k such that

$$(2.23) \quad z^k Y(\Phi(x_1)u, z) \Phi(x_2)v \in V[x_1^{\pm 1}, x_2^{\pm 1}, z].$$

Then

$$(z_1 - z_2)^k [Y_W(\Phi(x_1)u, z_1), Y_W(\Phi(x_2)v, z_2)] = 0,$$

which yields

$$(2.24) \quad (x_1^N - x_2^N)^k [\tilde{Y}_W(u, x_1), \tilde{Y}_W(v, x_2)] = 0.$$

Let r be a nonnegative integer such that $x^r \Phi(x)u \in V[x]$. Then

$$(x_0 + x_2)^r \Phi(x_0 + x_2)u = (x_2 + x_0)^r \Phi(x_2 + x_0)u \in V[x_0, x_2].$$

Furthermore, let $w \in W$. In view of Remark 2.5, we have

$$\begin{aligned} & (x_0 + x_2)^r \tilde{Y}_W(u, x_0 + x_2) \tilde{Y}_W(v, x_2)w \\ &= (x_0 + x_2)^r Y_W(\Phi(x_0 + x_2)u, (x_0 + x_2)^N) Y_W(\Phi(x_2)v, x_2^N)w \\ &= ((x_0 + x_2)^r Y_W(\Phi(x_0 + x_2)u, z_0 + x_2^N) Y_W(\Phi(x_2)v, x_2^N)w) \\ & \quad \big|_{z_0=(x_0+x_2)^N-x_2^N, x_0 \gg x_2} \end{aligned}$$

and

$$\begin{aligned} & (x_2 + x_0)^r \tilde{Y}_W(Y(u, x_0)v, x_2)w \\ &= (x_2 + x_0)^r Y_W(\Phi(x_2)Y(u, x_0)v, x_2^N)w \\ &= (x_2 + x_0)^r Y_W(Y(\Phi(x_2 + x_0)u, (x_2 + x_0)^N - x_2^N)\Phi(x_2)v, x_2^N)w \\ &= ((x_2 + x_0)^r Y_W(Y(\Phi(x_2 + x_0)u, z_0)\Phi(x_2)v, x_2^N)w) \big|_{z_0=(x_2+x_0)^N-x_2^N, x_2 \gg x_0}, \end{aligned}$$

using (2.16).

Assume $u \in V^j$ with $0 \leq j \leq N-1$. There exists a positive integer l such that

$$\begin{aligned} & (z_0 + x_2^N)^{l-\frac{j}{N}} Y_W((x_0 + x_2)^r \Phi(x_0 + x_2)u, z_0 + x_2^N) Y_W(\Phi(x_2)v, x_2^N)w \\ &= (x_2^N + z_0)^{l-\frac{j}{N}} Y_W(Y((x_0 + x_2)^r \Phi(x_0 + x_2)u, z_0)\Phi(x_2)v, x_2^N)w, \end{aligned}$$

which gives

$$(2.25) \quad \begin{aligned} & z_0^k (z_0 + x_2^N)^{l-\frac{j}{N}} Y_W((x_0 + x_2)^r \Phi(x_0 + x_2)u, z_0 + x_2^N) Y_W(\Phi(x_2)v, x_2^N)w \\ &= z_0^k (x_2^N + z_0)^{l-\frac{j}{N}} Y_W(Y((x_0 + x_2)^r \Phi(x_0 + x_2)u, z_0)\Phi(x_2)v, x_2^N)w, \end{aligned}$$

where k is the nonnegative integer as in (2.23). Now, we shall perform the substitution $z_0 = (x_2 + x_0)^N - x_2^N, x_0 \gg x_2$ on both sides. Notice that the expression on the right-hand side involves nonnegative integral powers of z_0 only, so that the substitutions $z_0 = (x_2 + x_0)^N - x_2^N, x_0 \gg x_2$ and $z_0 = (x_2 + x_0)^N - x_2^N, x_2 \gg x_0$ agree on the right-hand side. Performing the substitution $z_0 = (x_2 + x_0)^N - x_2^N, x_0 \gg x_2$ on both sides and using Remark 2.5, we obtain

$$\begin{aligned} & (x_0 + x_2)^{r+Nl-j} ((x_0 + x_2)^N - x_2^N)^k \cdot \\ & \quad \cdot Y_W(\Phi(x_0 + x_2)u, (x_0 + x_2)^N) Y_W(\Phi(x_2)v, x_2^N)w \\ &= \left((x_2 + x_0)^r (x_2^N + z_0)^{l-\frac{j}{N}} z_0^k Y_W(Y(\Phi(x_2 + x_0)u, z_0)\Phi(x_2)v, x_2^N)w \right) \\ & \quad \big|_{z_0=(x_2+x_0)^N-x_2^N, x_0 \gg x_2} \\ &= \left((x_2 + x_0)^r (x_2^N + z_0)^{l-\frac{j}{N}} z_0^k Y_W(Y(\Phi(x_2 + x_0)u, z_0)\Phi(x_2)v, x_2^N)w \right) \\ & \quad \big|_{z_0=(x_2+x_0)^N-x_2^N, x_2 \gg x_0} \\ &= (x_2 + x_0)^{r+Nl-j} ((x_2 + x_0)^N - x_2^N)^k \tilde{Y}_W(Y(u, x_0)v, x_2)w. \end{aligned}$$

Thus

$$(2.26) \quad \begin{aligned} & (x_0 + x_2)^{r+Nl-j} ((x_0 + x_2)^N - x_2^N)^k \tilde{Y}_W(u, x_0 + x_2) \tilde{Y}_W(v, x_2)w \\ &= (x_0 + x_2)^{r+Nl-j} ((x_0 + x_2)^N - x_2^N)^k \tilde{Y}_W(Y(u, x_0)v, x_2)w. \end{aligned}$$

Combining (2.24) and (2.26) we obtain the following quasi Jacobi identity

$$\begin{aligned}
 & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (x_1^N - x_2^N)^k \tilde{Y}_W(u, x_1) \tilde{Y}_W(v, x_2) w \\
 & \quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) (x_1^N - x_2^N)^k \tilde{Y}_W(v, x_2) \tilde{Y}_W(u, x_1) w \\
 (2.27) \quad & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (x_1^N - x_2^N)^k \tilde{Y}_W(Y(u, x_0)v, x_2) w.
 \end{aligned}$$

Therefore, (W, \tilde{Y}_W) carries the structure of a (G, ϕ) -quasi V -module. \square

REMARK 2.8. Let (W, Y_W) be any weak V -module. Then the same proof of Theorem 2.7 shows that (W, \tilde{Y}_W) is a (G, ϕ) -quasi V -module.

Next we present the second half of our main result of this paper.

THEOREM 2.9. *Let (W, Y_W) be a (G, ϕ) -quasi V -module. For $u \in V$, as in [BDM] set*

$$(2.28) \quad \bar{Y}_W(u, x) = Y_W(\Delta_N(x)u, x^{\frac{1}{N}}) \in (\text{End } W)[[x^{\frac{1}{N}}, x^{-\frac{1}{N}}]].$$

Then (W, \bar{Y}_W) carries the structure of a weak σ -twisted V -module.

PROOF. For convenience, let us simply use $\Delta(x)$ for $\Delta_N(x)$ in the proof. First, for $u \in V$, as $\Delta(x)u \in V[x^{1/N}, x^{-1/N}]$, we have

$$\bar{Y}_W(u, x) = Y_W(\Delta(x)u, x^{\frac{1}{N}}) \in \text{Hom}(W, W((x^{\frac{1}{N}}))).$$

Second,

$$\bar{Y}_W(\mathbf{1}, x) = Y_W(\Delta(x)\mathbf{1}, x^{\frac{1}{N}}) = Y_W(\mathbf{1}, x^{\frac{1}{N}}) = 1_W.$$

Third, for $u \in V$, as $\phi(\sigma) = \omega_N$, we have $Y_W(\omega_N^{-L(0)}\sigma(u), x) = Y_W(u, \omega_N x)$. Using Lemma 2.6, we get

$$\begin{aligned}
 \bar{Y}_W(\sigma(u), x) &= Y_W(\Delta(x)\sigma(u), x^{\frac{1}{N}}) \\
 &= Y_W(\omega_N^{-L(0)}\sigma\omega_N^{L(0)}\Delta(x)u, x^{\frac{1}{N}}) \\
 &= \lim_{z^{\frac{1}{N}} \rightarrow \omega_N x^{\frac{1}{N}}} Y_W(\omega_N^{L(0)}\Delta(x)u, z^{\frac{1}{N}}) \\
 &= \lim_{x^{\frac{1}{N}} \rightarrow \omega_N x^{\frac{1}{N}}} Y_W(\Delta(x)u, x^{\frac{1}{N}}) \\
 (2.29) \quad &= \lim_{x^{\frac{1}{N}} \rightarrow \omega_N x^{\frac{1}{N}}} \bar{Y}_W(u, x).
 \end{aligned}$$

Next, we prove the weak commutativity and the weak associativity, which amount to the twisted Jacobi identity. Let $u, v \in V$. As $\Delta(x)u, \Delta(x)v \in V[x^{1/N}, x^{-1/N}]$, from the quasi Jacobi identity there exists a nonnegative integer k such that

$$\begin{aligned}
 & z_0^{-1} \delta \left(\frac{x_1^{1/N} - x_2^{1/N}}{z_0} \right) (x_1 - x_2)^k \bar{Y}_W(u, x_1) \bar{Y}_W(v, x_2) \\
 & \quad - z_0^{-1} \delta \left(\frac{x_2^{1/N} - x_1^{1/N}}{-z_0} \right) (x_1 - x_2)^k \bar{Y}_W(v, x_2) \bar{Y}_W(u, x_1) \\
 (2.30) \quad & = x_1^{-1/N} \delta \left(\frac{x_2^{1/N} + z_0}{x_1^{1/N}} \right) (x_1 - x_2)^k Y_W(Y(\Delta(x_1)u, z_0)\Delta(x_2)v, x_2^{1/N}).
 \end{aligned}$$

It follows that there exists a nonnegative integer $k' \geq k$ such that

$$(2.31) \quad (x_1 - x_2)^{k'} [\bar{Y}_W(u, x_1), \bar{Y}_W(v, x_2)] = 0.$$

Next, we establish the weak associativity. Let $w \in W$. Assume $u \in V^j$ for some $0 \leq j \leq N-1$, i.e., $\sigma(u) = \omega_N^j u$. From (2.29), we have

$$x^{-j/N} \bar{Y}_W(u, x) \in (\text{End } W)[[x, x^{-1}]].$$

Let l be a nonnegative integer such that

$$x^{l-j/N} \bar{Y}_W(u, x) w \in W[[x]].$$

Using the commutation relation (2.31) we get

$$(2.32) \quad x_1^{l-j/N} (x_1 - x_2)^{k'} \bar{Y}_W(u, x_1) \bar{Y}_W(v, x_2) w \in W[[x_1, x_2^{1/N}]] [x_2^{-1/N}].$$

With (2.31), from (2.30) we get

$$\begin{aligned} & x_1^{-1/N} \delta \left(\frac{x_2^{1/N} + z_0}{x_1^{1/N}} \right) (x_1 - x_2)^{k'} Y_W(Y(\Delta(x_1)u, z_0) \Delta(x_2)v, x_2^{1/N}) w \\ &= \left(z_0^{-1} \delta \left(\frac{x_1^{1/N} - x_2^{1/N}}{z_0} \right) - z_0^{-1} \delta \left(\frac{x_2^{1/N} - x_1^{1/N}}{-z_0} \right) \right) \\ & \quad \cdot ((x_1 - x_2)^{k'} \bar{Y}_W(u, x_1) \bar{Y}_W(v, x_2) w) \\ &= x_1^{-1/N} \delta \left(\frac{x_2^{1/N} + z_0}{x_1^{1/N}} \right) ((x_1 - x_2)^{k'} \bar{Y}_W(u, x_1) \bar{Y}_W(v, x_2) w). \end{aligned}$$

In view of Remark 2.4, substituting $z_0 = (x_2 + x_0)^{1/N} - x_2^{1/N}$, $x_2 \gg x_0$ we get

$$\begin{aligned} & x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) \\ & \quad \cdot (x_1 - x_2)^{k'} Y_W(Y(\Delta(x_1)u, (x_2 + x_0)^{1/N} - x_2^{1/N}) \Delta(x_2)v, x_2^{1/N}) w \\ (2.33) \quad &= x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) ((x_1 - x_2)^{k'} \bar{Y}_W(u, x_1) \bar{Y}_W(v, x_2) w). \end{aligned}$$

For the expression on the left-hand side, using Proposition 2.3, we have

$$\begin{aligned} & x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) x_1^{l-j/N} (x_1 - x_2)^{k'} \\ & \quad \cdot Y_W(Y(\Delta(x_1)u, (x_2 + x_0)^{1/N} - x_2^{1/N}) \Delta(x_2)v, x_2^{1/N}) w \\ &= x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) x_0^{k'} (x_2 + x_0)^{Nl-j} \\ & \quad \cdot Y_W(Y(\Delta(x_2 + x_0)u, (x_2 + x_0)^{1/N} - x_2^{1/N}) \Delta(x_2)v, x_2^{1/N}) w \\ &= x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) x_0^{k'} (x_2 + x_0)^{Nl-j} Y_W(\Delta(x_2)Y(u, x_0)v, x_2^{1/N}) w \\ &= x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) x_0^{k'} (x_2 + x_0)^{Nl-j} \bar{Y}_W(Y(u, x_0)v, x_2) w. \end{aligned}$$

Then

$$\begin{aligned}
& x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) x_0^{k'} (x_2 + x_0)^{Nl-j} \bar{Y}_W(Y(u, x_0)v, x_2)w \\
&= x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) \left(x_1^{l-j/N} (x_1 - x_2)^{k'} \bar{Y}_W(u, x_1) \bar{Y}_W(v, x_2)w \right) \\
&\quad \quad \quad |_{x_1=x_2+x_0} \\
&= x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) \left(x_1^{l-j/N} (x_1 - x_2)^{k'} \bar{Y}_W(u, x_1) \bar{Y}_W(v, x_2)w \right) \\
&\quad \quad \quad |_{x_1=x_0+x_2} \\
&= x_1^{-1/N} \delta \left(\frac{(x_2 + x_0)^{1/N}}{x_1^{1/N}} \right) \left((x_0 + x_2)^{l-j/N} x_0^{k'} \bar{Y}_W(u, x_0 + x_2) \bar{Y}_W(v, x_2)w \right).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& (x_0 + x_2)^{l-j/N} x_0^{k'} \bar{Y}_W(u, x_0 + x_2) \bar{Y}_W(v, x_2)w \\
&= (x_2 + x_0)^{l-j/N} x_0^{k'} \bar{Y}_W(Y(u, x_0)v, x_2)w,
\end{aligned}$$

proving the weak associativity. Therefore, (W, \bar{Y}_W) is a weak σ -twisted V -module. \square

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